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On the mean motion induced by internal gravity waves

By FRANCIS P. BRETHERTON†
Institute of Geophysics and Planetary Physics,
University of California, La Jolla

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A train of internal gravity waves in a stratified liquid exerts a stress on the liquid, and induces changes in the mean motion of second order in the wave amplitude. In those circumstances in which the concept of a slowly varying quasi-sinusoidal wave train is consistent, the mean velocity is almost horizontal, and is determined to a first approximation irrespective of the vertical forces exerted by the waves. The sum of the mean flow kinetic energy and the wave energy is then conserved. The circulation around a horizontal circuit moving with the mean velocity is increased in the presence of waves according to a simple formula. The flow pattern is obtained around two- and three-dimensional wave packets propagating into a liquid at rest, and the results are generalized for any basic state of motion in which the internal Froude number is small. Momentum can be associated with a wave packet equal to the horizontal wave-number times the wave energy divided by the intrinsic frequency.

1. Introduction

Propagating waves in any medium normally transfer both energy and momentum. The concept of wave energy and its flux are well known [for a discussion in an arbitrary dispersive medium see Bretherton & Garrett (1968)], but the role in a general theory of wave momentum and radiation stress is still obscure. The momentum carried by a quantum of electromagnetic radiation, and the pressure exerted by an intense beam of it, are very familiar. Radiation pressure has observable consequences in other contexts. For example, the change in mean sea level due to swell breaking on a seashore has been discussed by Longuet-Higgins & Stewart (1962). In general, the mean stress exerted by a wave on a material medium is a tensor, and in particular cases it is normally straightforward to evaluate it for a sinusoidal propagating disturbance in terms of the wave-number, frequency and the square of the amplitude a. At least for surface waves on water (Longuet-Higgins & Stewart 1961), sound waves and Alven waves (Garrett 1968), the work done by this stress against the mean velocity accounts satisfactorily for the changes in *wave* energy for a wave train propagating in a shear flow. Such changes must be consistent with conservation of wave action (Bretherton & Garrett 1968). However, a general method of derivation of these relations is not yet available.‡ It is also not clear how far the radiation stress can be

† On leave from Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England.
‡ Progress has been made with this and will be reported elsewhere.

ascribed to the transfer with the group velocity of momentum carried by the wave, nor how that momentum is embodied in associated mean motions of order a^2. Tantalizingly suggestive results may be obtained for special cases, but apparent exceptions are also available.

Some of the difficulties which can arise are illustrated by internal gravity waves. The equations of motion for a non-dissipative Boussinesq liquid in which the density is $\rho(\mathbf{x}, t)$ may be written

$$\frac{D}{Dt}\mathbf{u}+\nabla p+\sigma\mathbf{n} = 0, \tag{1.1}$$

$$\nabla.\mathbf{u} = 0, \tag{1.2}$$

$$\frac{D\sigma}{Dt} = 0, \tag{1.3}$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t}+\mathbf{u}.\nabla,$$

$\mathbf{n}$ is unit vector vertically upwards, σ is a buoyancy force defined as $g(\rho-\rho^*)/\rho^*$, where ρ^* is some mean reference density (the inertial density) hereafter taken as unity, and $\rho^* p$ is the difference between the true pressure and the hydrostatic value in a liquid of uniform density ρ^*.† Defining now Eulerian mean values $\overline{\mathbf{u}}$, $\overline{p}$, $\overline{\sigma}$ and small fluctuations with zero mean $\mathbf{u}_1$, p_1, σ_1, the equations of mean motion are

$$\frac{D}{Dt}\overline{\mathbf{u}}+\nabla\overline{p}+\overline{\sigma}\mathbf{n} = -\nabla.\overline{\mathbf{u}_1\mathbf{u}_1}, \tag{1.4}$$

$$\nabla.\overline{\mathbf{u}} = 0, \tag{1.5}$$

$$\frac{D}{Dt}\overline{\sigma} = -\nabla.\overline{\mathbf{u}_1\sigma_1}. \tag{1.6}$$

Thus, in addition to the Reynolds stress tensor $\overline{\mathbf{u}_1\mathbf{u}_1}$ acting as a surface force on the mean flow, there are also changes in the mean buoyancy $\overline{\sigma}$.

For a slowly varying wave train (in which the amplitude a, wave-number $\mathbf{k}$ and frequency ω vary only slightly over a wavelength and period), the divergence of the Reynolds stress may be evaluated approximately by inserting at each point in space the value of $\overline{\mathbf{u}_1\mathbf{u}_1}$ computed *locally* as if the wave were strictly sinusoidal. The resulting divergence is then proportional to gradients of a^2, $\mathbf{k}$, ω. The buoy-

† The use of the Boussinesq approximation can sometimes give misleading results, even when the fractional range of density, δ, within the region of interest is very small. Long (1965) and Benjamin (1966) have shown how the structure and existence of solitary and cnoidal waves of arbitrary large horizontal wavelength λ are seriously altered if the variations of inertia of the fluid are ignored. However, the vertical scale of such waves is much smaller than the horizontal scale, and their structure is governed by a balance between the slight variations of phase velocity with wavelength and the small non-linearities associated with the amplitude a not being infinitesimal. It is then perhaps not surprising if the conceptual process of holding variations in density fixed while $\lambda \to \infty$, $a^2 \to 0$ and then allowing $\delta \to 0$ does not yield the same result as letting $\delta \to 0$ at the outset. In the present problem no such delicate balance is involved, and the same results may be obtained by retaining the full density variations till the penultimate step, provided the vertical scale of the wave packet is small compared with the density scale height. This is best treated within the framework of the general theory of radiation stress, referred to in the previous footnote.

ancy flux $\overline{\mathbf{u}_1\sigma_1}$, on the other hand, vanishes for a uniform train of internal gravity waves, because fluctuations of velocity and density are precisely out of phase. However, for a slightly non-uniform wave there is a small flux, and it is easily seen that the *body forces* due to the associated change in $\overline{\sigma}$ are comparable to the divergence of the *surface forces* $\overline{\mathbf{u}_1\mathbf{u}_1}$. From detailed calculations it appears that such changes in $\overline{\sigma}$ cannot in general be related to the instantaneous state of the wave train. Although $\nabla.\overline{\mathbf{u}_1\sigma_1}$ may be expressed in terms of the second derivatives of a^2, $\mathbf{k}$, ω, equation (1.6) is not integrable. Thus at first sight it would seem that the effect of the waves on the mean flow cannot be summarized as a stress alone, but must include a body force which depends in a complex manner both on the present state and on the past history of the waves.

This statement is overpessimistic. It will appear that the role of the buoyancy flux and the vertically acting Reynolds stresses is dynamically secondary, as they are offset by changes in density associated with very small vertical mean displacements. This conclusion follows from a consideration of the space and time scales of the mean motion implicit in the concept of a slowly varying wave train, and is associated with the statement that the internal Froude number is very much less than unity. Qualitative restrictions must be placed on the mean velocities which can be permitted, namely that they are effectively horizontal and determined independently at different levels. Then for most purposes a knowledge of the horizontally acting stresses is sufficient, and several general results follow concerning the form of the radiation stress tensor, conservation of energy, Kelvin's circulation theorem, and the momentum associated with a wave packet. The latter is the horizontal part of $\mathbf{k}/(\omega-\mathbf{u}.\mathbf{k})$ times the wave energy, although it may be realized as mean motion of the fluid distributed over considerable distances from the wave packet.

2. A quasi-sinusoidal wave train

2.1. *Linearized plane waves*

We are concerned with the additional motion of order a^2 induced by a wave in a medium of which the basic velocity is $\mathbf{u}_0(\mathbf{x},t)$, so that

$$\left.\begin{aligned}\overline{\mathbf{u}} &= \mathbf{u}_0+\overline{\mathbf{u}}_2+O(a^3),\\ \overline{\sigma} &= \sigma_0+\overline{\sigma}_2+O(a^3).\end{aligned}\right\} \tag{2.1}$$

An important special case is when the medium is basically at rest with uniform stratification

$$\mathbf{u}_0 = 0, \quad \nabla\sigma_0 = -N^2\mathbf{n}, \tag{2.2}$$

where the Brunt–Väisälä frequency N is constant. When the equations of motion (1.1)–(1.3) are linearized about the basic state (2.2), they have solutions in the form of plane transverse sinusoidal waves of frequency

$$\omega = N^{(0)}|\mathbf{k}\times\mathbf{n}|/|\mathbf{k}|. \tag{2.3}$$

The superscript (0) has been attached to N to make these formulas formally consistent with the scaled variables which will be introduced in §2.2. In the meantime, it may be ignored. The group velocity $\mathbf{c}$ is in the plane of the wave front in the direction of greatest slope and is inversely proportional to $|\mathbf{k}|$. Taking Cartesian axes so that Oz is vertical and $\mathbf{k}$ lies in the Oxz plane, we have

$$\mathbf{k} = (k, 0, m), \quad \omega = N^{(0)}|k|/\sqrt{(k^2+m^2)} \tag{2.4}$$

and
$$\mathbf{c} = (\alpha, \beta, \gamma) = N^{(0)}\frac{m}{(k^2+m^2)^{\frac{3}{2}}}\operatorname{sgn}k\,(m, 0, -k), \tag{2.5}$$

where $\operatorname{sgn} k = \pm 1$ according as $k \gtrless 0$. The wave energy density is

$$E = \tfrac{1}{2}\overline{\mathbf{u}_1^2} + \tfrac{1}{2}(1/N^2)\overline{\sigma_1^2}, \tag{2.6}$$

where the average indicated by the overbar is over the ensemble of realizations of the wave train subject to arbitrary changes of initial phase. The Reynolds stress tensor becomes

$$\mathsf{R} = \overline{\mathbf{u}_1\mathbf{u}_1} = \frac{E}{k^2+m^2}\begin{bmatrix} m^2 & 0 & -km \\ 0 & 0 & 0 \\ -km & 0 & k^2 \end{bmatrix}, \tag{2.7}$$

whereas the buoyancy flux vanishes

$$\mathbf{S} = \overline{\sigma_1\mathbf{u}_1} = 0. \tag{2.8}$$

The first two rows R_h of (2.7) describe the horizontal force exerted by the Reynolds stress. Comparison with (2.4) and (2.5) shows that they may be written

$$\mathsf{R}_h = \frac{E}{\omega}\mathbf{k}_h\,\mathbf{c}, \tag{2.9}$$

where $\mathbf{k}_h$ is a two element column vector describing the horizontal part of $\mathbf{k}$. The same is not true of the third row, which is

$$\mathsf{R}_v = -\frac{k^2}{m}\frac{E}{\omega}\,\mathbf{n}\mathbf{c}. \tag{2.10}$$

Also, if $\mathbf{u}_0$ is not zero but uniform, (2.3)–(2.9) continue to hold exactly, provided the frequency ω and group velocity $\mathbf{c}$ relative to a fixed observer are replaced by the *intrinsic* frequency and group velocity

$$\omega^+ = \omega - \mathbf{u}_0.\mathbf{k}, \quad \mathbf{c}^+ = \mathbf{c} - \mathbf{u}_0. \tag{2.11}$$

2.2. *Scale analysis*

If the basic state is not uniform, strictly plane waves are no longer possible, but we may consider a quasi-sinusoidal wave train, in which there is at each point a fairly well-defined dominant amplitude and wave-number, which, however, vary from place to place and time to time on scales much larger than those for the wave (a wavelength and wave period divided by 2π). The arguments of this paper are clarified if we introduce here a formal expansion in terms of a small parameter ϵ. We will suppose that the vertical, horizontal and group velocity scales for the wave train (H, L and C respectively) are order unity, whereas the wavelength and wave period are $O(\epsilon)$. Implicit also is the assumption that the slope k/m of the wave fronts is H/L and is order unity, although the argument is easily modified (without altering the conclusions), when this ratio is small. We set

$$\left.\begin{aligned} \mathbf{u}_1 &= \mathscr{R}\{a(\hat{\mathbf{u}}_1^{(0)} + \epsilon\hat{\mathbf{u}}_1^{(0)} + \ldots)\exp[i(\epsilon^{-1}\theta + \delta)]\}, \\ \sigma_1 &= \mathscr{R}\{a\epsilon^{-1}(\hat{\sigma}_1^{(0)} + \epsilon\hat{\sigma}_1^{(1)} + \ldots)\exp[i(\epsilon^{-1}\theta + \delta)]\}, \end{aligned}\right\} \quad (2.12)$$

where the amplitude $a(\mathbf{x}, t)$ and the phase function $\theta(\mathbf{x}, t)$ vary on scale unity. δ is an arbitrary additive constant in the phase. The lowest-order structure functions $\hat{\mathbf{u}}_1^{(0)}(\mathbf{x}, t)$, $\hat{\sigma}_1^{(0)}(\mathbf{x}, t)$ are those appropriate to a strictly sinusoidal wave in a uniform medium defined by the local values of $\mathbf{u}_0$, N^2. Spatial gradients of $\mathbf{u}_1$, σ_1 are dominated by the term $\epsilon^{-1}\theta$ in the exponent of (2.12), and they are $O(\epsilon^{-1})$ larger than $\mathbf{u}_1$, σ_1 themselves. The local frequency and wave-number are thus

$$\epsilon^{-1}\omega = -\epsilon^{-1}(\partial\theta/\partial t), \quad \epsilon^{-1}\mathbf{k} = \epsilon^{-1}\nabla\theta, \quad (2.13)$$

respectively.

This picture implies that the Brunt–Väisälä frequency N (which is comparable to the time frequency $\epsilon^{-1}\omega$ of the wave) must be $O(\epsilon^{-1})$, although the space and time scales of $\mathbf{u}_0$ are at least those of the wave train. Thus we are forced to conclude that, if the concept of a slowly varying wave-train of internal gravity waves is to be consistent, the internal Froude number $|\mathbf{u}_0|^2/N^2H^2$ of the basic state must be very much less than unity. This implies important qualitative restrictions on the permissible velocities $\mathbf{u}_0$, which we will now investigate. To avoid excessive verbiage, we will envisage the horizontal, vertical and time scales for $\mathbf{u}_0$ as H, L and H/C respectively, the same as those for the wave train. This implies $|\mathbf{u}_0|/C \leqslant O(1)$. For some applications the scales may be larger than this and for safety the argument should be reworked, but in most cases the conclusions are strengthened.

Taking the curl of equation (1.4) we see that $\nabla\sigma \times \mathbf{n}$ is order unity in ϵ, even though $\nabla\sigma_0$ itself must by $O(\epsilon^{-2})$. Thus

$$\sigma_0 = \sigma_0^{(0)}(z) + \epsilon^2\sigma_0^{(2)}(\mathbf{x}, t) + \ldots \quad (2.14)$$

and from (1.6) where

$$\mathbf{u}_0 = \mathbf{u}_0^{(0)}(\mathbf{x}, t) + \epsilon^2\mathbf{u}_0^{(2)}(\mathbf{x}, t)\ldots,$$

we have

$$w_0^{(0)} = \mathbf{u}_0^{(0)} . \mathbf{n} = 0. \quad (2.15)$$

Thus the vertical velocities in the basic flow must be two orders of magnitude smaller in ϵ than the horizontal ones, and the latter (by (1.5)) must be quasi-non-divergent. Expanding also

$$p_0 = \epsilon^{-2}\{p_0^{(0)}(z) + \epsilon^2 p_0^{(2)}(\mathbf{x}, t)\ldots\} \quad (2.16)$$

the equations of motion (1.4)–(1.6) for the basic state reduce to

$$(D_0/Dt)\mathbf{u}_0 + \nabla_h p_0^{(2)} = 0, \quad (2.17)$$

$$\frac{dp_0^{(0)}}{dz} = -\sigma_0^{(0)}(z), \quad \frac{\partial p_0^{(2)}}{\partial z} = -\sigma_0^{(2)}(\mathbf{x}, t), \quad (2.18)$$

$$\nabla_h . \mathbf{u}_0^{(0)} = 0, \quad (2.19)$$

$$(D_0/Dt)\sigma_0^{(2)} = -N^{(0)2}w_0^{(2)}, \quad (2.20)$$

where $\nabla_h = \nabla - \mathbf{n}(\partial/\partial z)$ is the horizontal part of the operator ∇.

Inspection of (2.17)–(2.20) shows (2.17) and (2.19) (with appropriate boundary conditions) form a closed set independent of (2.18) and (2.20). The horizontal motions at different levels are essentially decoupled, the pressure perturbation $p_0^{(2)}$ being determined separately at each level as a consequence of conservation of vorticity about a vertical axis. Then (2.18) determines $\sigma_0^{(2)}$, and (2.20) $w_0^{(2)}$. This dynamical régime is a necessary consequence of the whole concept of a slowly varying train of internal gravity waves and must be continually borne in mind in the sequel.

With the definitions (2.13) and the expansion (2.14) for σ_0, it is easily seen that, correct to lowest-order in ϵ, the velocities (2.12) satisfy the linearized equations of motion for plane waves and the dispersion relation (2.3) with ω replaced by ω^+ according to (2.10). At each point $\mathbf{x}, t$ the local values of ω^+, $\mathbf{k}$, $N^{(0)}(z)$ are appropriate. The factor ϵ^{-1} in σ_1 ensures that $\hat{\sigma}_1^{(0)}$ is formally of order unity. It should be noted that the Brunt–Väisälä frequency $N^{(0)}$ is a function of z only. It is essentially inconsistent to suppose that N^2 depends on horizontal position on a scale comparable to the vertical scale H of the basic motion.

One loophole does remain in the argument leading to (2.15), pointed out by Garrett (1968). $\sigma_0^{(0)}(z)$ in (2.14) could be a function of time, implying vertical velocities $w_0^{(0)}(z,t)$ which are horizontally uniform and requiring a uniform horizontal convergence to match. In an unbounded fluid, this convergence would imply $\mathbf{u}_0 \gg C$ at large distances, violating the assumed scales on ω, $\mathbf{k}$. Nevertheless, moving vertical boundaries could be imposed to preserve consistency. This somewhat pathological case is considered in §4.2.

2.3. *Second-order mean motions*

Associated with the first-order perturbation velocities (2.12) are second-order ones

$$\left.\begin{aligned} \mathbf{u}_2 &= \mathscr{R}\{a^2(\hat{\mathbf{u}}_2^{(0)}+\epsilon\hat{\mathbf{u}}_2^{(1)}+\ldots)\exp[2i(\epsilon^{-1}\theta+\delta)]\}+\overline{\mathbf{u}}_2, \\ \sigma_2 &= \mathscr{R}\{a^2\epsilon^{-1}(\hat{\sigma}_2^{(0)}+\epsilon\hat{\sigma}_2^{(1)}+\ldots)\exp[2i(\epsilon^{-1}\theta+\delta)]\}+\overline{\sigma}_2. \end{aligned}\right\} \quad (2.21)$$

The sinusoidal term here is the first harmonic of the fundamental oscillation, forced locally by quadratic non-linearities (for the transverse waves considered here it happens that $\hat{\mathbf{u}}_2^{(0)}$ vanishes identically). $\overline{\mathbf{u}}_2(\mathbf{x},t)$ on the other hand, is the slowly varying second-order mean velocity which is of primary interest to us. It is the average of $\mathbf{u}_2$ formed by integrating over a period with respect to initial δ and dividing by 2π. It may also be regarded with an error smaller than any power of ϵ, as the running space or time average formed by weighting $\mathbf{u}_2$ with some suitable normalized infinitely differentiable 'window' function F which vanishes for arguments outside a certain range

$$\overline{\mathbf{u}}_2(\mathbf{x}) = \int F\{\epsilon^{-\frac{1}{2}}(\mathbf{x}-\mathbf{x}')\}\mathbf{u}_2(\mathbf{x}')\,d\mathbf{x}' \Big/ \int F\{\epsilon^{-\frac{1}{2}}(\mathbf{x}-\mathbf{x}')\}\,d\mathbf{x}'. \quad (2.22)$$

Equations (1.4)–(1.6) now become, to lowest-order in ϵ

$$\frac{D_0}{Dt}\overline{\mathbf{u}}_2+(\overline{\mathbf{u}}_2\,.\,\nabla)\,\mathbf{u}_0+\nabla\overline{p}_2+\overline{\sigma}_2\,\mathbf{n} = -\nabla\,.\,\mathsf{R}, \quad (2.23)$$

$$\nabla\,.\,\overline{\mathbf{u}}_2 = 0, \quad (2.24)$$

$$\frac{D_0}{Dt}\overline{\sigma}_2-\epsilon^{-2}N^{(0)2}(z)\overline{w}_2 = -\nabla\,.\,\mathbf{S}, \quad (2.25)$$

where the buoyancy force

$$\mathbf{S} = \epsilon^{-1}\overline{\mathbf{u}_1^{(1)}\sigma_1^{(1)}} + \overline{\{\mathbf{u}_1^{(2)}\sigma_1^{(1)} + \mathbf{u}_1^{(1)}\sigma_1^{(2)}\}} + O(a^2\epsilon) \tag{2.26}$$

is at first sight formally of order ϵ^{-1}. However $\mathbf{u}_1^{(1)}$, $\sigma_1^{(1)}$ have precisely the structure appropriate to strictly sinusoidal waves, so by (2.8) the first term vanishes. The wave energy density E is, with the present scaling, also $O(a^2\epsilon^0)$, so the right-hand side of (2.23)–(2.25) is everywhere of this order.

The appearance in (2.25) of the factor ϵ^{-2}, shows that, when w_2 is expanded in powers of ϵ,

$$\overline{w}_2^{(0)} = \overline{\mathbf{u}}_2^{(0)}\,.\,\mathbf{n} = 0. \tag{2.27}$$

Thus the forced motion $\overline{\mathbf{u}}_2$ is subject to precisely similar dynamical restrictions as the basic flow, i.e. it is essentially horizontal, with the flow at different levels independently determined, and the small vertical velocities just enough to ensure continual hydrostatic balance. Only those components R_h of the Reynolds stress tensor which can exert a force in the horizontal direction are of real significance, and using (2.9), the dominant equations governing the forced second-order mean motion are

$$\frac{D_0}{Dt}\overline{\mathbf{u}}_2^{(0)} + (\overline{\mathbf{u}}_2^{(0)}\,.\,\nabla)\,\mathbf{u}_0 + \nabla_h p_2^{(2)} = -\nabla\,.\left\{\frac{E}{\omega^+}\mathbf{k}_h\,\mathbf{c}^+\right\}, \tag{2.28}$$

$$\nabla\,.\,\overline{\mathbf{u}}_2^{(0)} = 0. \tag{2.29}$$

These form a closed set. $\overline{\sigma}_2^{(2)}$, $\overline{w}_2^{(2)}$ follow from

$$\overline{\sigma}_2^{(2)} = -\frac{\partial \overline{p_2^{(2)}}}{\partial z} + \nabla\,.\left\{\frac{k^2}{m}\frac{E}{\omega^+}\,\mathbf{c}^+\right\} \tag{2.30}$$

and

$$\overline{w}_2^{(2)} = \frac{1}{N^{(0)2}}\left\{\frac{D_0}{Dt}\overline{\sigma}_2^{(2)} - \nabla\,.\,\mathbf{S}\right\}, \tag{2.31}$$

but these equations are essentially secondary, only their orders of magnitude being necessary to verify the consistency of (2.28) and (2.29). The latter will form the basis of our subsequent development, although to avoid needless repetition the superscripts will henceforth be omitted.

2.4. *Kelvin's circulation theorem*

We now derive an important first integral of (2.28). If we consider fluid particles moving horizontally with the total mean velocity $\mathbf{u}_0 + \overline{\mathbf{u}}_2$, the circulation round a circuit Γ moving with the same velocity is

$$C = \oint_\Gamma (\mathbf{u}_0 + \overline{\mathbf{u}}_2)\,.\,d\mathbf{s}. \tag{2.32}$$

Then

$$\begin{aligned}\frac{dC}{dt} &= \oint_\Gamma [\{(D_0/Dt)(\mathbf{u}_0+\overline{\mathbf{u}}_2) + (\mathbf{u}_2\,.\,\nabla)\,\mathbf{u}_0\}\,.\,d\mathbf{s} + (\mathbf{u}_0+\overline{\mathbf{u}}_2)\,.\,\{(d\mathbf{s}\,.\,\nabla)(\mathbf{u}_0+\overline{\mathbf{u}}_2)\}] \\ &= \oint_\Gamma \left[\left\{\frac{D_0}{Dt}u_{0i} + u_{0j}\frac{\partial u_{0j}}{\partial x_i}\right\}ds_i + \left\{\frac{D_0}{Dt}\overline{u}_{2i} + \overline{u}_{2j}\frac{\partial}{\partial x_j}u_{0i} + \overline{u}_{2j}\frac{\partial u_{0j}}{\partial x_i} + u_{0j}\frac{\partial \overline{u}_{2j}}{\partial x_i}\right\}ds_i\right] \\ &= -\oint \frac{\partial}{\partial x_j}\left\{\frac{E}{\omega^+}k_i(c-u_0)_j\right\}ds_i,\end{aligned} \tag{2.33}$$

where the basic equation of motion

$$(D_0/Dt)\,\mathbf{u}_0 + \nabla_h\, p_0 = 0$$

has been used to eliminate the $O(a^0)$ terms, and the fact that the integral of a gradient round a closed circuit vanishes. All terms of order a^4 have been omitted.

Now it was shown by Bretherton & Garrett (1968) that when a quasi-sinusoidal wave train moves in a slowly varying medium the quantity E/ω^+ is conserved, i.e.

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega^+}\right) + \nabla\,.\left(\frac{E}{\omega^+}\mathbf{c}\right) = 0. \tag{2.34}$$

It is the complete group velocity $\mathbf{c}$ which enters (2.34), not the intrinsic value $\mathbf{c}^+ = \mathbf{c} - \mathbf{u}_0$. Thus

$$\frac{\partial}{\partial x_j}\left\{\frac{E}{\omega^+}(c - u_0)_j\right\} = -\frac{D_0}{Dt}\left(\frac{E}{\omega^+}\right). \tag{2.35}$$

Also the wave-number $\mathbf{k}$ varies in a systematic manner, according to the spatial gradients of the parameters N, $\mathbf{u}_0$ which enter the dispersion relation. Because N is independent of horizontal position,

$$(\partial/\partial t)\,\mathbf{k}_h + (\mathbf{c}\,.\,\nabla)\,\mathbf{k}_h = -(\nabla\mathbf{u}_0)\,.\,\mathbf{k}_h, \tag{2.36}$$

so that

$$(c_j - u_{0j})\frac{\partial k_i}{\partial x_j} = -\frac{D_0}{Dt}k_i - k_j\frac{\partial u_{0j}}{\partial x_i}. \tag{2.37}$$

From (2.33), (2.35) and (2.37), and remembering that to a sufficient approximation for the right-hand side

$$\frac{D}{Dt}(ds) = \frac{D}{Dt}(ds_i) = ds_j\frac{\partial u_{0i}}{\partial x_j}, \tag{2.38}$$

we have

$$\frac{dC}{dt} = \frac{d}{dt}\oint_\Gamma \mathbf{k}_h\frac{E}{\omega^+}\,.\,d\mathbf{s},$$

i.e.

$$C = C_0 + \oint_\Gamma \mathbf{k}_h\frac{E}{\omega^+}\,.\,d\mathbf{s}, \tag{2.39}$$

where C_0 is a constant, the circulation associated with the basic flow before the waves arrived there.

Equation (2.39) is a statement of Kelvin's circulation theorem, as applied to the mean flow driven by the waves but disregarding the motion of fluid particles on the scale of the waves themselves. Taken with (2.19) and (2.29), it completely fixes the mean motion. It may be derived also from the constancy of circulation round a circuit consisting of real material particles, all of the same density. Originally horizontal, when waves arrive the circuit is distorted on scale $O(\epsilon^{-1})$ and the velocities are also altered, but it may be shown directly that, correct to order $a^2\epsilon^0$, the extra circulation over and above that due to the horizontal mean velocities round the horizontal projection of the time circuit is

$$-\oint \mathbf{k}_h\frac{E}{\omega^+}\,.\,d\mathbf{s}.$$

2.5. *Conservation of energy*

If we consider a horizontal flow $\mathbf{u}_0+\overline{\mathbf{u}}_2$ satisfying (2.28) and (2.29) *exactly*, it is easy to derive the equation, correct to order a^2,

$$\begin{aligned}(\partial/\partial t)\{\tfrac{1}{2}\,|\,\mathbf{u}_0+\overline{\mathbf{u}}_2\,|^2\}+\nabla\,.\,\{(\mathbf{u}_0+\overline{\mathbf{u}}_2)(\tfrac{1}{2}\,|\,\mathbf{u}_0+\overline{\mathbf{u}}_2\,|^2+p_0^{(2)}+\overline{p_2^{(2)}})\}\\ = -\nabla\,.\,\{\mathbf{u}_0\,.\,\mathsf{R}_h\}+\nabla\mathbf{u}_0:\mathsf{R}_h\\ = -\frac{\partial}{\partial x_j}\left\{\frac{E}{\omega^+}\,k_i u_{0i}c_j^+\right\}+\frac{\partial u_{0i}}{\partial x_j}\frac{E}{\omega^+}\,k_i c_j^+.\end{aligned} \tag{2.40}$$

The second term on the right-hand side is the inner product of the rate of strain tensor with the Reynolds stress tensor. Now since the dispersion relation

$$\omega = \mathbf{u}_0\,.\,\mathbf{k}+N^{(0)}\{|\,\mathbf{n}\times\mathbf{k}|/|\,\mathbf{k}\,|\}, \tag{2.41}$$

explicitly involves time and horizontal position only through $\mathbf{u}_0(\mathbf{x},t)$, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathbf{c}\,.\,\nabla\right)(\omega-\mathbf{u}_0\,.\,\mathbf{k}) &= \left(\frac{\partial\omega}{\partial t}\right)_{\mathbf{k}}+\mathbf{u}_0\,.\,(\nabla\omega)_{\mathbf{k}}-\mathbf{k}\,.\left(\frac{\partial}{\partial t}+\mathbf{c}\,.\,\nabla\right)\mathbf{u}_0\\ &= k_i\frac{\partial u_{0i}}{\partial t}+u_j\frac{\partial u_{0i}}{\partial x_j}k_i-k_i\left(\frac{\partial}{\partial t}+c_j\frac{\partial}{\partial x_j}\right)u_{0i}\\ &= -(\partial u_{0i}/\partial x_j)\,k_i c_j^+.\end{aligned} \tag{2.42}$$

Thus the last term on the right-hand side of (2.40) may be written

$$-\frac{E}{\omega^+}\left(\frac{\partial}{\partial t}+\mathbf{c}\,.\,\nabla\right)\omega^+ = -\left\{\frac{\partial}{\partial t}E+\nabla\,.\,(cE)\right\}+\omega^+\left\{\frac{\partial}{\partial t}\left(\frac{E}{\omega^+}\right)+\nabla\,.\left(\mathbf{c}\frac{E}{\omega^+}\right)\right\}$$

and we have

$$\begin{aligned}&\frac{\partial}{\partial t}\{\tfrac{1}{2}\,|\,\mathbf{u}_0+\overline{\mathbf{u}}_2\,|^2+E\}\\ &\quad+\nabla\,.\left\{(\mathbf{u}_0+\overline{\mathbf{u}}_2)(\tfrac{1}{2}\,|\,\mathbf{u}_0+\overline{\mathbf{u}}_2\,|^2+E+p_0^{(2)}+p_2^{(2)})+\frac{\omega}{\omega^+}\,\mathbf{c}^+E\right\} = 0.\end{aligned} \tag{2.43}$$

This expresses conservation of total energy, allowing for an interchange between the wave and the mean flow. The apparent flux of wave energy relative to the local fluid is

$$(\omega/\omega^+)\,E\mathbf{c}^+,$$

rather than $E\mathbf{c}^+$. The difference $(\mathbf{u}_0\,.\,\mathbf{k})(E/\omega^+)\,\mathbf{c}^+$ is accounted for by the flux of wave momentum in a velocity $\mathbf{u}_0$. It should be noted that there is no potential energy term in (2.43) corresponding to vertical displacements of order ϵ^2 associated with the mean flow in the stratification of order ϵ^{-2}. Although changes in potential energy are implied by higher-order approximations to (2.28) and (2.29), these are not available for conversion into wave energy or kinetic energy of mean motion, and they do not figure in the total energy as computed this way.

3. The mean motion induced by wave packets

3.1. *Wave packets*

A wave packet is a quasi-sinusoidal wave train of which the amplitude $a(\mathbf{x},t)$ is negligibly small everywhere outside a moving volume V across which the frequency and wave-number may be regarded as effectively uniform. Viewed from the scale of variation of the medium, the wave packet may be regarded as a point, but it still contains enough wavelengths for the analysis of the previous section to be applicable.

However, this definition at once raises a difficulty because the scales for the variation of amplitude are now smaller [say $O(\epsilon^{\frac{1}{2}})$] than those of the frequency and wave-number and other properties of the basic state. Thus, strictly speaking, the previous analysis only tells us how the packet moves and drives the mean flow for times only as long as it takes to move its own diameter, i.e. $O(\epsilon^{\frac{1}{2}})$, whereas we really need to be able to follow it for a time of order unity. That this point is not trivial follows from noting that small differences $[O(\epsilon^{\frac{1}{2}})]$ of group velocity across the packet can cause after time unity a radical redistribution of wave energy within the volume V, so that the packet does not propagate without change of shape. However, it is plausible that the total wave energy within V varies in proportion to the intrinsic frequency ω^+, consistent with conservation of wave action (2.34), and the mean motions associated with the packet may still be computed over this time scale (2.28) and (2.29). Nevertheless, to verify these statements requires a detailed analysis of error terms which have been dismissed in §2.

This problem disappears if we do not try to consider an individual wave packet on its own, but rather regard it as a brick out of which continuous wave trains are constructed by juxtaposition. Equations (2.28), (2.29) and (2.34) are linear in $\bar{\mathbf{u}}_2$ and E, and the right-hand side has the form of a divergence. Thus, granted these equations, we may analyze them *mathematically* by regarding $E(\mathbf{x}, t)$ as the superposition

$$E(\mathbf{x}) = \int E(\mathbf{x}')\,\delta(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}'$$

of localized parts $E(\mathbf{x}')\,\delta(\mathbf{x}-\mathbf{x}')$, each of which moves according to (2.34) at the local group velocity with constant E/ω^+, and each of which is surrounded by a forced velocity field $\bar{\mathbf{u}}_2$ which, by (2.39) is irrotational and non-divergent everywhere except at $\mathbf{x} = \mathbf{x}'$. To avoid confusion, the wave energy will not be regarded as concentrated precisely at a point, but as distributed over a very small volume V. The value of this approach is in the conceptual simplification of regarding a complicated wave train as made up of a lot of particles, each moving individually according to well defined laws. It is not necessary to justify the approximate equations in detail for each particle, but only for their superposition.

3.2. *The flow round a three-dimensional wave packet*

The nature of the mean flow induced by a wave packet in a stationary medium follows at once from the remarks that it is entirely horizontal, non-divergent, and according to the circulation theorem (2.39) irrotational except where the wave energy density E is non-zero. Thus at each level separately it is the two-dimensional incompressible motion around a patch of vorticity concentrated at the point $\mathbf{x}'(t)$, and is instantaneously determined by the distribution of E. At large distances the velocities fall off as the inverse square, as the field due to a two-dimensional dipole, and the integral of $\overline{\mathbf{u}}_2$ over the whole horizontal plane is formally divergent. Nevertheless, there is a conversed quantity, the impulse, defined as half the dipole moment of the vorticity distribution (Batchelor 1967), which can justifiably be interpreted as the momentum associated with the patch. At each level, it is equal to the horizontal integral of $\mathbf{k}_h E/\omega^+$, and equals the resultant of the external forces which would be required, if applied locally within V, to start the motion $\overline{\mathbf{u}}_2(\mathbf{x})$ impulsively from rest. As the wave packet propagates upwards through the fluid, the impulse at each level changes in sympathy, but the total integrated over all levels is constant. Thus, the total momentum associated with the wave packet is $\mathbf{k}_h/\omega^+$ times the total wave energy.

When the basic velocity $\mathbf{u}_0$ does not vanish, the circulation in the absence of the wave packet will not, in general, vanish. But if the volume V occupied by the wave packet is small compared to the scale of variations of $\mathbf{u}_0$, it is clear that the flow around the packet approximates to that when $\mathbf{u}_0$ is uniform. Viewed from the scale of $\mathbf{u}_0$, it appears singular at the point $\mathbf{x}'$, where there is a dipole of strength

$$2\int \mathbf{k}_h \frac{E}{\omega^+}\,dx\,dy.$$

This dipole moves with the group velocity $\mathbf{c}$ appropriate to the wave-number $\mathbf{k}$, and it may also fluctuate slowly in strength as $\mathbf{k}_h$ varies. The motions induced by this dipole throughout the flow $\mathbf{u}_0(\mathbf{x}, t)$ depend in a complicated manner on the distribution of basic vorticity $\mathbf{n}.\nabla\times\mathbf{u}_0$ (Lighthill 1957), but the effect of the wave packet is entirely summed up by its 'momentum'

$$\frac{\mathbf{k}_h}{\omega^+}\int E\,dV. \tag{3.1}$$

When this varies as the packet moves, it appears that the total momentum of the mean flow varies in some sense in sympathy.

It is instructive also to introduce the possibility of some internal friction in the wave packet, so that (2.34) should be replaced by

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega^+}\right)+\nabla.\left(\mathbf{c}\frac{E}{\omega^+}\right) = -\frac{1}{\omega^+}D, \tag{3.2}$$

where D is the dissipation rate per unit volume, but if the logarithmic decrement $D/(E\epsilon^{-1}\omega)$ is small the other propagation characteristics and the mean flow will not be directly affected. Molecular viscosity would provide a suitable mechanism. Reworking (2.35)–(2.39), we now have

$$\frac{dC}{dt} = \frac{d}{dt}\oint\frac{\mathbf{k}_h}{\omega^+}E\,.\,d\mathbf{s}+\oint\frac{\mathbf{k}_h}{\omega^+}D\,.\,d\mathbf{s}. \tag{3.3}$$

The last term describes mean circulation which can be permanently induced even in a uniform medium, when a passing wave packet is partially dissipated. A distributed residual of dipole moments is left after the packet is past, and the momentum earmarked to the dissipated part of the packet is transferred to the mean flow.

3.3. *Radiation by a two-dimensional wave packet in a medium at rest*

If the whole flow field is two-dimensional (i.e. independent of y) the second-order mean motions induced by a wave train have quite different character. $\mathbf{k}_h$ and $\bar{\mathbf{u}}_2^{(0)}$ are everywhere parallel to Ox, and Kelvin's circulation theorem (2.39) is trivially satisfied, each side vanishing identically for a horizontal circuit Γ. The only non-divergent one-dimensional velocity $\bar{\mathbf{u}}_2$ is independent of x and presumably zero so the scale analysis leading to equations (2.28) and (2.29) must break down somewhere.

Near the wave packet there will be a balance

$$\frac{\partial}{\partial x}\overline{p_2^{(2)}} = -\nabla\cdot\left(\frac{E}{\omega^+}k\mathbf{c}^+\right),$$
$$= -\frac{\partial}{\partial x}\left(k\frac{E}{\omega^+}\alpha\right) - \frac{\partial}{\partial z}\left(k\frac{E}{\omega^+}\gamma\right). \tag{3.4}$$

and, as the horizontal integral of the right-hand side does not vanish, there is a pressure difference

$$\overline{p_2^{(2)}}\Big|_{-\infty}^{+\infty} = -\frac{\partial}{\partial z}\left\{\gamma\left(\int_{-\infty}^{\infty}k\frac{E}{\omega^+}dx\right)\right\} \tag{3.5}$$

across a two-dimensional wave packet. $\pm\infty$ in this context means sufficiently far for the wave energy density E to be effectively zero. According to (2.30) and (2.31), small vertical velocities $\epsilon^2\overline{w_2^{(2)}}$ are inevitably associated with changes in the pressure $\overline{p_2^{(2)}}$, and with the vertical forces and buoyancy flux in the wave packet, implying small horizontal convergences $\epsilon^2(\partial\overline{u}_2^{(2)}/\partial x)$. The associated horizontal velocities are indeed negligibly small, *unless the convergence extends over a very large area.* But this is exactly what is implied by (3.5), with $\partial\overline{p}_2^{(2)}/\partial z$ uniform and non-zero for an apparently infinite distance on either side of the wave packet.

Clearly we must reconsider equations (2.28) and (2.29) when we are looking at motions distance $O(\epsilon^{-1}H)$ from the wave packet, for on this scale the stratification is no longer sufficiently strong to enforce quasi-non-divergent horizontal motion. The wave packet has shrunk to a line of vertical dimension H, moving vertically with velocity γ. Only the horizontal integral

$$P(z,t) = \int_V k\frac{E}{\omega^+}dx \tag{3.6}$$

is significant in forcing the mean motion, which is as if a horizontal body force $(\partial/\partial z)\{\gamma P(z-\gamma t)\}$ about the Oy axis concentrated along $x = 0$ were applied to an otherwise undisturbed horizontally stratified mean flow. This causes a quasi-hydrostatic system of radiating internal gravity waves, of vertical scale H and timescale H/γ, and hence of horizontal scale $\epsilon^{-1}N_0^{(0)}H^2/\gamma$. The induced velocities are $O(a^2\epsilon)$ and nearly horizontal, the pressure and density fields everywhere being in hydrostatic balance. It is not easy to justify formally the omission of the vertical forces exerted by the wave packet and the buoyancy flux, but, unlike the horizontal forces they are easily negated by small localized vertical displacements, and the picture presented here appears to be plausible.

We will illustrate by considering the motion at large distances from a wave packet propagating in a uniform medium at rest (N^2 = constant, $\mathbf{u}_0 = 0$), so that the vertical component of group velocity γ is constant, and the wave energy distribution can propagate upwards without change of shape. We will assume the latter to be Gaussian

$$E(x,z,t) = \frac{A}{\pi LH} \exp - \left\{ \frac{(x-\alpha t)^2}{L^2} + \frac{(z-\gamma t)^2}{H^2} \right\}. \tag{3.7}$$

We introduce a scaled horizontal co-ordinate

$$\tilde{x} = \epsilon x \tag{3.8}$$

and a stream function $\overline{\psi}_2^{(1)}$,

$$\overline{u}_2 = \epsilon \frac{\partial \overline{\psi}_2^{(1)}}{\partial z}, \quad \overline{w}_2 = -\frac{\partial}{\partial x}(\epsilon \overline{\psi}_2^{(1)}) = -\epsilon^2 \frac{\partial}{\partial \tilde{x}} \overline{\psi}_2^{(1)}. \tag{3.9}$$

In terms of $\tilde{x}$ the linearized equation of motion is

$$\frac{\partial^2}{\partial t^2} \left\{ \frac{\partial^2 \overline{\psi}_2^{(1)}}{\partial z^2} \right\} + N^{(0)2} \frac{\partial^2 \overline{\psi}^{(1)}}{\partial \tilde{x}^2} = \frac{\partial^2}{\partial t^2} \left(\frac{\partial P}{\partial z} \right) \delta(\tilde{x}), \tag{3.10}$$

where a term

$$\epsilon^2 \frac{\partial^2}{\partial t^2} \frac{\partial^2 \overline{\psi}_2^{(1)}}{\partial \tilde{x}^2}$$

has been omitted, thus assuming hydrostatic balance, and

$$\int_{-\infty}^{\infty} \delta(\tilde{x})\, dx = \epsilon^{-1}.$$

We also have

$$P = P(z - \gamma t). \tag{3.11}$$

Relative to axes moving vertically with velocity γ, (3.10) and (3.11) have a steady formal solution

$$\overline{\psi}_2^{(1)} = \iint_{-\infty}^{\infty} d\tilde{\kappa}\, d\mu \frac{i\gamma^2 \mu^3 \hat{P}}{\mu^4 \gamma^2 - N^{(0)2} \tilde{\kappa}^2} \exp\{i(\tilde{\kappa}\tilde{x} + \mu z)\}, \tag{3.12}$$

where $\hat{P}(\tilde{\kappa}, \mu)$ is the two-dimensional Fourier transform of $P(\tilde{x}, z)$,

$$\hat{P} = \frac{1}{4\pi^2} \frac{Ak}{\omega} \exp\{-\tfrac{1}{4}\mu^2 H^2\}. \tag{3.13}$$

The three-dimensional axisymmetric wave pattern produced by a disturbance moving vertically in a uniformly stratified liquid has been studied in detail by Warren (1960), Lighthill (1967) and Mowbray & Rarity (1967), using methods similar to the present one, though without making the hydrostatic approximation. Some discussion of the two-dimensional case has also been given by Rarity (1967). These authors have shown how, at large distances from the source, the integral they obtained corresponding to (3.12) is dominated by those wavenumbers $\epsilon\tilde{\kappa}$, μ for which the denominator vanishes. In a nearly horizontal direction in the x, z plane the integrand is small unless

$$\epsilon\tilde{\kappa}/\mu \sim z/x, \tag{3.14}$$

in which case it approximates to (3.12). Care must be taken to avoid the singularity in the way corresponding to an outgoing wave, by taking the denominator as

$$(\gamma\mu + i\sigma)^2\mu^2 - \tilde{\kappa}^2 N^{(0)2},$$

where σ is small and positive. This is equivalent to letting $\tilde{\kappa}$ be complex, having a small imaginary part with sign the same as μ near $\tilde{\kappa} = -\gamma\mu^2/N^{(0)}$, but with opposite sign near $+\gamma\mu^2/N^{(0)}$.

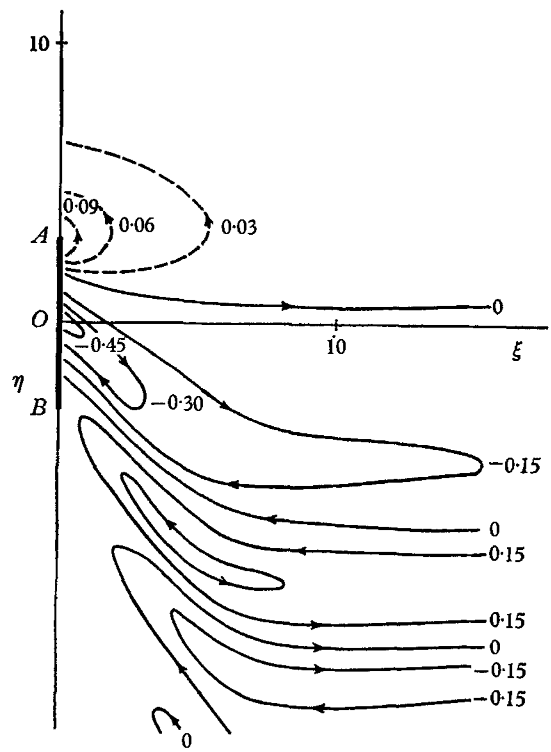

FIGURE 1. One-half of the symmetrical wake behind a moving wave packet, as given by contours of

$$I(\xi, \eta) = -\int_0^\infty \lambda \exp(-\lambda^2) \cos(\xi\lambda^2 + \zeta\lambda)\, d\lambda.$$

The line AB shows approximately the region occupied by the packet, the energy density being proportional to $\exp(-\tfrac{1}{4}\zeta^2)$ and concentrated on $\xi = 0$.

Now (3.12) may be integrated immediately with respect to κ, and using (3.13) then with respect to μ. We have

$$\overline{\psi_2^{(1)}} = \frac{2Ak}{\pi N^{(0)} H^2 \omega} I\left(\frac{|x|}{NH^2}, \frac{z}{H}\right), \tag{3.15}$$

where
$$I(\xi, \zeta) = -\int_0^\infty \lambda \exp(-\lambda^2) \cos(\xi\lambda^2 + \zeta\lambda)\, d\lambda. \tag{3.16}$$

Contours of $\overline{\psi_2^{(1)}}(\tilde{x}, z)$ are shown in figure 1. The abscissa is scaled by a factor

$2\gamma/N^{(0)}H$ relative to the ordinate. 90% of the wave energy lies within the region $|\xi| \sim 0, |\eta| < 3$, indicated by the vertical line near the origin.

$$\bar{u}_2 = \epsilon \frac{\partial \overline{\psi}_2^{(1)}}{\partial z} = \frac{\epsilon\gamma}{N^{(0)}H} \frac{2}{\pi} \frac{Ak}{H^2 w} \frac{\partial I}{\partial \zeta} \tag{3.17}$$

is non-zero in the vicinity of the wave packet, but $\overline{w}^{(2)}$ is discontinuous across it. As $\tilde{x} \to 0+$ we have, after some manipulation

$$\left.\frac{\partial \overline{\psi_2^{(1)}}}{\partial \tilde{x}}\right|_{0+} = -\frac{\gamma}{2N^{(0)2}} \frac{\partial^2 P}{\partial z^2}, \tag{3.18}$$

with an equal magnitude but opposite sign as $\tilde{x} \to 0-$. This shows that there is indeed a pressure drop across the packet of magnitude given by (3.5).

Also from (3.17) we may obtain

$$\int_{-\infty}^{\infty} \epsilon \frac{\partial \overline{\psi_2^{(1)}}}{\partial z} dx = P(z - \gamma t) \tag{3.19}$$

showing that at each level the wave momentum is indeed related to the wave energy as in §3.2. However, now the horizontal velocities are $O(\epsilon)$ everywhere, and the momentum resides predominantly in the radiation field at large distances rather than near the wave packet. When $\tilde{x} = O(\epsilon)$, (3.12) cannot be used to determine $\overline{\psi}_2$, which is $O(\epsilon^2)$ there. The vertical forces must be taken into account but the additional velocities are essentially of order $a^2\epsilon^2$, whereas the total induced horizontal velocity is $O(a^2\epsilon)$, (3.17).

The essential difference between the two- and three-dimensional cases is that in the former all the forces exerted by the packet on the mean flow can be balanced (locally at least) by changes in pressure which involve only small vertical displacements in the strong basic stratification. In the latter, the rotational part of the horizontal forces causes velocities $O(a^2\epsilon^2)$ directly in the vicinity of the wave packet.

4. Other considerations

4.1. *A general argument*

The association of a momentum equal to k/ω^+ times the wave energy with a wave packet is strongly suggested by an argument due to Professor R. W. Stewart, to whom the author is very grateful for permission to reproduce it here.

Consider a wave packet being generated by a system of external forces f of order a, moving rigidly parallel to Ox relative to a frame of reference in which the fluid is basically at rest with the intrinsic phase velocity

$$c_p^+ = \omega^+/k \tag{4.1}$$

appropriate to that direction. Such a system of forces could be exerted by a wavy rigid boundary

$$z = h(x, t) = a\epsilon \exp\{-(x - c_p^+ t)^2/L^2\} \sin \epsilon^{-1} k(x - c_p^+ t). \tag{4.2}$$

The forces are everywhere normal to the boundary (in a perfect fluid) and may thus have a resultant F of order a^2 in the Ox direction. This force F may indeed be calculated by asserting that the rate of working to move the boundary is exactly

$$W = Fc_p^+, \tag{4.3}$$

and in a conservative system this must be equal to the rate of increase of energy of the wave packet. Now, in a frame of reference in which the basic velocity $\mathbf{u}_0$ vanishes, we may estimate changes in wave energy by multiplying the sinusoidal external forces f (of order a) by the perturbation velocities (also of order a), and from a linearized theory alone we obtain

$$W = \frac{\partial}{\partial t}\int_V E\,d\mathbf{x}, \tag{4.4}$$

where E is the wave energy density in the usual (external) sense (Eckart 1960; Bretherton & Garrett 1968), and the integration is over the volume of fluid in which waves have been significantly excited. From (4.3) and (4.4)

$$F = \frac{W}{c_p^+} = \frac{k}{\omega^+}\frac{\partial}{\partial t}\int_V E\,d\mathbf{x}. \tag{4.5}$$

Thus there is a resultant force exerted by the boundary parallel to itself, equal to the component of wave-number in that direction, divided by the intrinsic frequency, times the rate of supply of wave energy, and unless the reaction of the fluid is such as to negate it, the momentum of the fluid must increase at the same rate.

This argument is very general, and could apparently be applied to all kinds of mechanical systems, provided it is possible to imagine a wave packet generated in a conservative manner by a rigidly moving system of forces. It may sometimes be used to relate the momentum transfer to the wave energy flux in the interior of a system. Nevertheless, this apparent generality could be misleading if each individual case is not looked at carefully. The reasoning depends on a system of forces moving relative to the fluid in such a manner that the energy transferred to the fluid is equal to the work done by the agency moving the forces. Wind-generated waves on the surface of water, for example, clearly do not satisfy this condition, neither do transverse body waves in an elastic solid. It appears to be necessary that there be no tangential stress on the boundary across which the forces are applied, or no forces in the direction of their motion if they are applied in the interior. Furthermore, if we are to estimate the momentum transfer in the interior of a perfect fluid, before the fluid on one side of a wavy surface may be replaced in imagination by a rigid body, we must be sure that it deviates from plane *everywhere* by no more than $O(a^2)$, and that there is no mass transport across it (even to $O(a^2)$). Otherwise the zero-order normal stress has to be included in the reckoning.

For internal gravity waves, profound and subtle questions arise if we attempt to apply the argument to surfaces moving in directions other than horizontal. Gradients of basic pressure along the mean position of the surface make it difficult to average quantities unambiguously over a wavelength. If second-order horizontal mean velocities are really intrinsic to trains of internal gravity waves we cannot be sure what forces are occasioned when they are not permitted to cross our specified surface. Finally, any momentum imparted to the fluid in the vertical direction is rapidly counteracted by buoyancy forces, so it never appears as particle motion. Nevertheless, the argument appears to be correct for wave momentum in a horizontal direction, and its elegance demands further research into the precise conditions for its validity.

4.2. *The exceptional case*

Finally, we consider the case proposed by Garrett (1968), in which the basic vertical velocities w_0 are of order unity, but with error $O(\epsilon^2)$ are independent of horizontal position. $N^{(0)2}$ is still horizontally uniform, satisfying

$$\frac{D^{(0)}}{Dt} N^{(0)2} = -N^{(0)2} \frac{\partial w_0^{(0)}}{\partial z}, \tag{4.6}$$

but as far as the induced mean velocities $\bar{\mathbf{u}}_2$ are concerned, all the scaling considerations of §2 continue to hold. Conservation of wave action (2.34) is still valid, though ω will change with time in sympathy with $N^{(0)}$. Horizontal circuits Γ remain horizontal and Kelvin's circulation theorem (2.39) goes through as before. The only change comes when we consider conservation of energy (§2.5).

Remembering that at all times an arbitrary function of z, t may be subtracted from the buoyancy σ without affecting the dynamics, provided a corresponding function is subtracted from the pressure, the equations of motion for the complete mean velocities

$$\bar{\mathbf{u}} = \mathbf{u}_0 + \bar{\mathbf{u}}_2$$

are, to order $a^2\epsilon^0$,

$$\frac{D^{(0)}}{Dt} \overline{\mathbf{u}^{(0)}} + \overline{\sigma^{(2)}}\mathbf{n} + \nabla \overline{p^{(2)}} = -\nabla \cdot \left\{\frac{E}{\omega^+}\mathbf{k}_h \mathbf{c}^+\right\} + \mathbf{n}\nabla \cdot \left(\frac{k^2}{m}\frac{E}{\omega^+}\mathbf{c}^+\right), \tag{4.7}$$

$$\frac{D^{(0)}}{Dt} \overline{\sigma^{(2)}} - N^{(2)2}\overline{w^{(2)}} = -\nabla \cdot \mathbf{S}^{(0)}, \tag{4.8}$$

$$\nabla \cdot \overline{\mathbf{u}^{(0)}} = 0. \tag{4.9}$$

Multiplying (4.7) by $\overline{\mathbf{u}^{(0)}}$:

$$\frac{\partial}{\partial t}\{\tfrac{1}{2}\,|\overline{\mathbf{u}^{(0)}}|^2\} + \nabla \cdot \{\overline{\mathbf{u}^{(0)}}(\tfrac{1}{2}\,|\overline{\mathbf{u}^{(0)}}|^2 + p^{(2)})\} + \sigma^{(2)} w_0^{(0)}$$
$$= -\nabla \cdot \left\{\left(\mathbf{u}_h \mathbf{k}_h - \frac{k^2}{m} w_0^{(0)}\right)\frac{E}{\omega^+}\mathbf{c}^+\right\} + \nabla \mathbf{u}_h : \left(\mathbf{k}_h \frac{E}{\omega^+}\mathbf{c}^+\right) - \frac{\partial w_0^{(0)}}{\partial z}\frac{k^2}{m}\frac{E}{\omega^+}\gamma^+. \tag{4.10}$$

When manipulating the Reynolds stress terms we have to remember that $\mathbf{u}_0$ now has a vertical component and $N^{(0)}$ depends on time. Thus two additional terms arise in (2.42), one from $(\partial\omega/\partial t)_\mathbf{k}$ and one from $w_0(\partial\omega/\partial z)_\mathbf{k}$. Together they add

$$\frac{\omega^+}{N^{(0)}}\frac{D^{(0)}}{Dt}N^{(0)}$$

to the right-hand side of (2.42) and, using (4.6),

$$\nabla \mathbf{u}_h : \left(\frac{E}{\omega^+}\mathbf{k}_h\,\mathbf{c}^+\right) - \frac{\partial w_0}{\partial z}\frac{k^2}{m}\frac{E}{\omega^+}\gamma^+ = -\left\{\frac{\partial}{\partial t}E + \nabla.(\mathbf{c}E)\right\} + \tfrac{1}{2}E\frac{\partial w_0}{\partial z}. \qquad (4.11)$$

Thus two additional terms arise in the energy equation, a rate of working $\overline{\sigma}^{(2)}w_0$ by the mean buoyancy forces against the basic vertical velocities and the term $\frac{1}{2}E(\partial w_0/\partial z)$ on the right-hand side. There does not seem to be any simple way of combining these to give a conservation equation, for to estimate $\overline{\sigma}^{(2)}$ we need (4.8), which in turn involves $\mathbf{S}^{(0)}$, which must be computed from (2.26) and $\overline{w}^{(2)}$. Both these terms involve carrying the approximations in a quasi-sinusoidal wave train to a higher-order than so far considered. It may be possible to do so, but the algebra is complicated.

This situation need cause no concern. The second-order mean motion is still horizontal and completely determined by (2.28) and (2.29), and changes in wave energy follow from (2.39). The expansion of the complete energy in powers of ϵ apparently does not lead to an approximate equation involving only those quantities calculable from (2.29). However, this was also true in §2.5, only there the potential energy terms for the mean flow could be subtracted off identically and consistently ignored, even though they could not be estimated without going to a higher-order in ϵ. Given order unity vertical velocities in the basic state this does not appear to be possible, even in the complete absence of waves. This example illustrates particularly the ambiguous nature of energy density in all fluid motions in that the amount which is available for conversion into kinetic form depends on the constraints imposed. If the latter are in some sense approximate, an approximate form of the available energy may be appropriate, but there is no *a priori* guarantee that this be so, and in this case it appears that it is not. The primary conclusions of this paper are unimpaired.

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